

# On the minimal rank in non-reflexive operator spaces over finite fields

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June 16, 2014

## Abstract

Let  $U$  and  $V$  be vector spaces over a field  $\mathbb{K}$ , and  $\mathcal{S}$  be an  $n$ -dimensional linear subspace of  $\mathcal{L}(U, V)$ . The space  $\mathcal{S}$  is called algebraically reflexive whenever it contains every linear map  $g : U \rightarrow V$  such that, for all  $x \in U$ , there exists  $f \in \mathcal{S}$  with  $g(x) = f(x)$ . A theorem of Meshulam and Šemrl states that if  $\mathcal{S}$  is not algebraically reflexive then it contains a non-zero operator  $f$  of rank at most  $2n - 2$ , provided that  $\mathbb{K}$  has more than  $n + 2$  elements. In this article, we prove that the provision on the cardinality of the underlying field is unnecessary. To do so, we demonstrate that the above result holds for all finite fields.

*AMS Classification:* 15A03, 47L05.

*Keywords:* Algebraic reflexivity; Rank; Finite fields.

## 1 Introduction

Let  $\mathbb{K}$  be an arbitrary field and  $U$  and  $V$  be vector spaces over  $\mathbb{K}$ . Given a linear subspace  $\mathcal{S}$  of the space  $\mathcal{L}(U, V)$  of all linear maps from  $U$  to  $V$ , its reflexive closure is defined as

$$\mathcal{R}(\mathcal{S}) := \{g \in \mathcal{L}(U, V) : \forall x \in U, \exists f \in \mathcal{S} : g(x) = f(x)\};$$

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it is obviously a linear subspace of  $\mathcal{L}(U, V)$  that contains  $\mathcal{S}$ , and one checks that  $\mathcal{R}(\mathcal{S}) = \mathcal{R}(\mathcal{R}(\mathcal{S}))$ . One says that  $\mathcal{S}$  is **(algebraically) reflexive** whenever  $\mathcal{R}(\mathcal{S}) = \mathcal{S}$ .

An active research topic consists in finding sufficient conditions for the reflexivity of an operator space in terms of the dimension of  $\mathcal{S}$  and the rank of its elements. Denote by

$$\text{mrk}(\mathcal{S}) := \min\{\text{rk}(f) \mid f \in \mathcal{S} \setminus \{0\}\}$$

the minimal rank among the non-zero operators in  $\mathcal{S}$  (here we do not distinguish between infinite cardinals and simply write  $\text{rk}(f) = +\infty$  if  $f$  is not a finite rank operator).

Assume now that  $\mathcal{S}$  is non-reflexive and finite-dimensional. In [2], Larson showed that

$$\text{mrk}(\mathcal{S}) < +\infty.$$

Hence, a finite-dimensional operator space that contains no non-zero operator of finite rank is always reflexive. A natural improvement is to give an upper-bound for  $\text{mrk}(\mathcal{S})$  with respect to the dimension of  $\mathcal{S}$ . In [1], Ding showed that

$$\text{mrk}(\mathcal{S}) \leq (\dim \mathcal{S})^2.$$

Later, this upper-bound was substantially improved by Meshulam and Šemrl: in [5], they showed that

$$\#\mathbb{K} > \dim \mathcal{S} + 2 \Rightarrow \text{mrk}(\mathcal{S}) \leq 2 \dim \mathcal{S} - 2.$$

Earlier, this result had been obtained by Li and Pan for the field of complex numbers [3].

For 2-dimensional spaces, this upper bound is known to be optimal (see [5]). For algebraically closed fields, Meshulam and Šemrl further improved the upper-bound as follows in [6]:

$$\text{mrk}(\mathcal{S}) \leq \dim \mathcal{S}.$$

In [8], we examined whether the upper-bound  $2 \dim \mathcal{S} - 2$  from Meshulam and Šemrl's result was optimal or if one could improve it in the case when  $\dim \mathcal{S} \geq 3$ . First, it was proved that this upper-bound still held under the milder cardinality assumption  $\#\mathbb{K} > \dim \mathcal{S}$ , and then, under that provision, a classification of the non-reflexive  $n$ -dimensional operator spaces  $\mathcal{S}$  such that  $\text{mrk} \mathcal{S} = 2n - 2$  was

achieved (see Theorem 6.1 of [8]): it was shown in particular that the existence of such spaces is connected to the existence of exotic division algebra structures over the field  $\mathbb{K}$ , called left-division-bilinearizable (LDB) division algebras. The existence of LDB division algebras over  $\mathbb{K}$  is deeply connected to the quadratic structure of  $\mathbb{K}$ . LDB division algebras were entirely classified in [7], and as a consequence the following result was obtained<sup>1</sup>:

**Theorem 1.1.** *Let  $\mathcal{S}$  be a non-reflexive  $n$ -dimensional subspace of  $\mathcal{L}(U, V)$ , with  $\#\mathbb{K} > n \geq 3$ . If  $\mathbb{K}$  has characteristic not 2 and  $n \notin \{3, 5, 9\}$ , then*

$$\mathrm{mrk}(\mathcal{S}) \leq 2n - 3.$$

*If  $\mathbb{K}$  has characteristic 2 and  $n - 1$  is not a power of 2, then*

$$\mathrm{mrk}(\mathcal{S}) \leq 2n - 3.$$

For finite fields, this can even be improved as follows:

**Theorem 1.2.** *Let  $\mathcal{S}$  be a non-reflexive  $n$ -dimensional subspace of  $\mathcal{L}(U, V)$ , with  $\mathbb{K}$  a finite field such that  $\#\mathbb{K} > n \geq 3$ . Then,*

$$\mathrm{mrk}(\mathcal{S}) \leq 2n - 3.$$

This follows from Theorem 6.1 of [8] and from the fact, over a finite field, a quadratic form whose dimension is greater than 2 is always isotropic.

In this article, we consider the situation of small finite fields. Until now, the best known result over such fields was the following one:

**Proposition 1.3** (See Theorem 4.5 in [8]). *Let  $\mathcal{S}$  be an  $n$ -dimensional non-reflexive operator space. Then,*

$$\mathrm{mrk}(\mathcal{S}) \leq \frac{n(n+1)}{2}.$$

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<sup>1</sup>When  $U$  and  $V$  are finite-dimensional, Theorem 1.1 is a straightforward consequence of Theorem 6.1 of [8] and of Corollary 1.3 of [7]. To obtain the general case, it suffices to extend the former to all vector spaces  $U$  and  $V$ , which can be done by noticing that Meshulam and Šemrl's Corollary 2.5 of [5] states that if  $\mathrm{mrk}(\mathcal{S}) = 2 \dim \mathcal{S} - 2$  then all the operators in  $\mathcal{S}$  have finite-rank (provided that  $\#\mathbb{K} > n + 2$ , but it has been shown in [8] that it suffices to assume that  $\#\mathbb{K} > n$ ) and one can then simply apply the above results to the reduced space associated with  $\mathcal{S}$ , whose source and target spaces are finite-dimensional (see Section 3 for the definition of that reduced space).

Here, we shall improve this upper-bound as follows, thus generalizing Meshulam and Šemrl's theorem to all fields:

**Theorem 1.4.** *Let  $\mathcal{S}$  be an  $n$ -dimensional non-reflexive operator space. Then,*

$$\mathrm{mrk}(\mathcal{S}) \leq 2n - 2.$$

To achieve this, we will prove that Theorem 1.4 holds for all finite fields. In the case when the source space of the operators in  $\mathcal{S}$  is finite-dimensional, we will use counting techniques together with very basic results from linear algebra to obtain the above result (Section 2). These methods were inspired by an article of Meshulam and Šemrl [4], in which a similar technique was used to study locally linearly dependent spaces of operators over finite fields. In the last section, the general result will be derived from this situation by using a theorem of Larson [2].

Before we proceed with the proof of Theorem 1.4, we would like to make a few observations. First of all, Theorem 1.2 shows that  $2n - 2$  is not an optimal upper-bound for finite fields of large cardinality and  $n \geq 3$ . We do not know whether the upper bound  $2n - 3$  holds for arbitrary finite fields when  $n \geq 3$ . In any case, it is known that the optimal upper-bound must be greater than or equal to  $n$ , owing to the existence of  $n$ -dimensional division algebras over any finite field (see [6]). Our last remark is that results on non-reflexive spaces are often obtained as special cases of results on locally linearly dependent operator spaces. Recall that the subspace  $\mathcal{S} \subset \mathcal{L}(U, V)$  is called **locally linearly dependent** (in short: **LLD**) whenever every vector  $x \in U$  is annihilated by some operator  $f \in \mathcal{S} \setminus \{0\}$ . If  $\mathcal{S}$  is non-reflexive and one chooses  $g \in \mathcal{R}(\mathcal{S}) \setminus \mathcal{S}$ , then  $\mathcal{S} \oplus \mathbb{K}g$  is an LLD space of which  $\mathcal{S}$  is a linear hyperplane. Most of the results we have cited are actually special cases of results on linear hyperplanes of LLD spaces (provided that  $\dim \mathcal{S} \geq 2$ ; on the other hand, every 1-dimensional operator space is reflexive). However, it is not true that a linear hyperplane of an LLD space is always non-reflexive. One special feature of the proof of Theorem 1.2 is that we shall use the full power of the non-reflexivity assumption instead of relying only upon local linear dependence. We do not know whether the upper bound  $2n - 2$  in Theorem 1.4 holds for linear hyperplanes of locally linearly dependent spaces as well (with  $n \geq 2$ ).

## 2 Proof of Theorem 1.4 in the finite-dimensional setting over a finite field

Throughout this section, we assume that the field  $\mathbb{K}$  is finite and  $q$  denotes its cardinality. Let  $U$  and  $V$  be vector spaces over  $\mathbb{K}$ , and  $\mathcal{S}$  be a finite-dimensional non-reflexive subspace of  $\mathcal{L}(U, V)$ . Assume that  $U$  is finite-dimensional, set

$$p := \dim U \quad \text{and} \quad n := \dim \mathcal{S},$$

and assume that

$$\text{mrk}(\mathcal{S}) > 2n - 2,$$

so that

$$p \geq 2n - 1.$$

We seek to find a contradiction. Classically, if  $n \leq 1$  then  $\mathcal{S}$  would be reflexive and hence

$$n \geq 2.$$

Let us choose an operator  $g \in \mathcal{R}(\mathcal{S}) \setminus \mathcal{S}$ . Then,

$$\mathcal{T} := g + \mathcal{S}$$

is an  $n$ -dimensional affine subspace of  $\mathcal{L}(U, V)$  that does not contain 0, and the assumption  $g \in \mathcal{R}(\mathcal{S})$  translates into:

$$\forall x \in U, \exists h \in \mathcal{T} : h(x) = 0.$$

Let us consider the set

$$\mathcal{N} := \{(x, h) \in U \times \mathcal{T} : h(x) = 0\}.$$

**Claim 1.** *The affine space  $\mathcal{T}$  contains an operator  $h$  such that  $\text{rk } h \leq n - 1$ .*

*Proof.* Assume on the contrary that no such operator exists. For all  $x \in U$ , either  $x = 0$  and then all the operators  $h \in \mathcal{T}$  satisfy  $h(x) = 0$ , or  $x \neq 0$  and then at least one operator  $h \in \mathcal{T}$  satisfies  $h(x) = 0$ . This leads to

$$q^n + q^p - 1 \leq \#\mathcal{N}.$$

On the other hand, for every  $f \in \mathcal{T}$ , we have  $\dim \text{Ker } f \leq p - n$  and hence at most  $q^{p-n}$  vectors of  $U$  are annihilated by  $f$ . This yields

$$\#\mathcal{N} \leq q^n q^{p-n} = q^p.$$

Combining the above two inequalities leads to  $q^n - 1 \leq 0$ , contradicting  $n > 0$ .  $\square$

Now, let us set

$$r := \min\{\text{rk } h \mid h \in \mathcal{T}\}.$$

Note that we have just proved that

$$r \leq n - 1.$$

In the rest of the proof, we shall use the following simple remark: if there are distinct operators  $h_1$  and  $h_2$  in  $\mathcal{T}$  such that  $\text{rk } h_1 = r$  and  $\text{rk } h_2 \leq 2n - 2 - r$ , then  $h_1 - h_2$  is a non-zero operator of  $\mathcal{S}$  and

$$\text{rk}(h_1 - h_2) \leq r + (2n - 2 - r) = 2n - 2,$$

which contradicts our assumption that  $\text{mrk}(\mathcal{S}) > 2n - 2$ . Thus, we obtain:

**Claim 2.** *The space  $\mathcal{T}$  contains exactly one rank  $r$  operator, and all the other ones have their rank greater than  $2n - 2 - r$ .*

Next, we prove:

**Claim 3.** *One has  $r = n - 1$ .*

*Proof.* Recall from the proof of Claim 1 that  $q^n + q^p - 1 \leq \#\mathcal{N}$ . On the other hand, the sole rank  $r$  operator of  $\mathcal{T}$  annihilates exactly  $q^{p-r}$  vectors of  $U$ , whereas every other operator in  $\mathcal{T}$  annihilates at most  $q^{p-2n+1+r}$  vectors. This leads to

$$\#\mathcal{N} \leq q^{p-r} + (q^n - 1)q^{p-2n+1+r},$$

and hence

$$q^{p-r}(q^r - 1) \leq (q^n - 1)(q^{p-2n+1+r} - 1).$$

In particular, as  $r > 0$  we find  $p - 2n + 1 + r > 0$ , and factoring yields

$$q^{r-n+1} \geq \frac{1 - q^{-r}}{(1 - q^{-n})(1 - q^{2n-p-1-r})}.$$

Obviously, as  $q \geq 2$  and  $r > 0$ ,

$$\frac{1 - q^{-r}}{(1 - q^{-n})(1 - q^{2n-p-1-r})} > 1 - q^{-r} \geq \frac{1}{2},$$

and hence  $r - n + 1 > -1$ , which leads to  $r \geq n - 1$ . □

Now, we know that  $\mathcal{T}$  contains one rank  $n-1$  operator, which we denote by  $h_0$ , and all the other ones have greater rank. Set

$$m := \#\{f \in \mathcal{T} : \text{rk}(f) \leq n\},$$

so that  $m \leq q^n$  and  $\mathcal{T}$  contains exactly  $m-1$  rank  $n$  operators, and exactly  $q^n - m$  operators with rank greater than  $n$ . This leads to

$$\#\mathcal{N} \leq q^{p-n+1} + (m-1)q^{p-n} + (q^n - m)q^{p-n-1}. \quad (1)$$

For every  $h \in \mathcal{T}$  such that  $\text{rk } h = n$ , we have

$$\dim(\text{Ker } h \cap \text{Ker } h_0) \geq \dim \text{Ker } h + \dim \text{Ker } h_0 - \dim U = p - 2n + 1.$$

Thus, at least  $q^{p-2n+1}$  vectors of  $\text{Ker } h_0$  belong to  $\text{Ker } h$ . Considering the subset

$$\mathcal{N}' := \mathcal{N} \cap ((\text{Ker } h_0 \setminus \{0\}) \times (\mathcal{T} \setminus \{h_0\})),$$

this leads to

$$\#\mathcal{N}' \geq (q^{p-2n+1} - 1)(m - 1),$$

and hence

$$q^n + q^p - 1 + (q^{p-2n+1} - 1)(m - 1) \leq \#\mathcal{N}. \quad (2)$$

Combining (1) with (2) leads to

$$q^n + q^p - q^{p-2n+1} - q^{p-n+1} + q^{p-n} - q^{p-1} \leq m(q^{p-n} - q^{p-n-1} - q^{p-2n+1} + 1). \quad (3)$$

As  $q \geq 2$  we have on the other hand

$$q^{p-n} - q^{p-n-1} - q^{p-2n+1} + 1 \geq q^{p-n-1}(q - 1) - q^{p-2n+1} \geq q^{p-n-1} - q^{p-2n+1} \geq 0,$$

where the last inequality comes from  $n \geq 2$ . As  $m \leq q^n$  we deduce that

$$q^n + q^p - q^{p-2n+1} - q^{p-n+1} + q^{p-n} - q^{p-1} \leq q^n(q^{p-n} - q^{p-n-1} - q^{p-2n+1} + 1).$$

Expanding and simplifying leads to

$$q^{p-n} \leq q^{p-2n+1}.$$

Yet,  $q^{p-2n+1} < q^{p-n}$  since  $n \geq 2$ .

This final contradiction shows that our initial assumption was wrong. This yields

$$\text{mrk}(\mathcal{S}) \leq 2n - 2,$$

thereby completing the proof of Theorem 1.4 in the special case when  $\mathbb{K}$  is finite and the source space of  $\mathcal{S}$  is finite-dimensional.

### 3 The generalization to operator spaces between infinite-dimensional spaces

Now, we complete the proof of Theorem 1.4 for finite fields. Assume that  $\mathbb{K}$  is finite.

We lose no generality in assuming that  $\mathcal{S}$  is a minimal non-reflexive space. Then, by a theorem of Larson [2, Corollary 2.8], all the operators in  $\mathcal{S}$  have finite rank. It follows that

$$U_0 := \bigcap_{f \in \mathcal{S}} \text{Ker } f$$

has finite codimension in  $U$ . Then, every  $f \in \mathcal{R}(\mathcal{S})$  naturally induces a linear operator

$$\bar{f} : U/U_0 \rightarrow V$$

with the same rank as  $f$ , to the effect that the **reduced space**

$$\bar{\mathcal{S}} := \{\bar{f} \mid f \in \mathcal{S}\}$$

has dimension  $n$  and the vector space  $\overline{\mathcal{R}(\mathcal{S})}$  is isomorphic to  $\mathcal{R}(\mathcal{S})$ , whose dimension is greater than  $n$ . One checks that  $\overline{\mathcal{R}(\mathcal{S})} \subset \mathcal{R}(\bar{\mathcal{S}})$  (actually, those spaces are equal), and hence  $\bar{\mathcal{S}}$  is non-reflexive. Then, as  $U/U_0$  is finite-dimensional, we deduce from Section 2 that

$$\text{mrk}(\mathcal{S}) = \text{mrk}(\bar{\mathcal{S}}) \leq 2n - 2,$$

which completes the proof.

*Remark 3.1.* If  $\mathcal{S}$  contains an operator with infinite rank, then applying the above result to the subspace  $\mathcal{S}_F$  of all finite rank operators in  $\mathcal{S}$  - which, by a theorem of Larson [2], is non-reflexive - yields  $\text{mrk } \mathcal{S} = \text{mrk } \mathcal{S}_F \leq 2n - 4$ . Therefore, if  $\text{mrk } \mathcal{S} \geq 2n - 3$  then  $\mathcal{S}$  contains only finite rank operators.

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